Comparison of M-ary Modulation Systems

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Consideration of large alphabet digital communication systems is of both theoretical and practical interest. Although performance bounds on optimum systems for the Gaussian channel are available, constructive methods for approaching these bounds are unknown, except in a few very special cases. Specific systems have been proposed and evaluated relative to these bounds, but exact evaluation of error probability is generally a difficult numerical task. It is of interest to consider simpler performance criteria which permit comparison of various systems without extensive computation.

An easily evaluated criterion (based on the alphabet size and minimum distance between signal vectors) is shown to yield a simple sufficient condition for one system to be better than another (smaller error probability for the same energy-per-bit). The criterion is applied to orthogonal, biorthogonal, simplex, and more general permutation modulation systems. In addition to comparing the various systems, we consider ways of obtaining good special cases of permutation modulation. Finally, we assess a recently proposed system ("N-orthogonal phase modulation") and show that it is generally inferior to more conventional techniques.

I. INTRODUCTION

The choice of waveforms for communicating over the Gaussian additive noise channel is a classic problem in communication theory. Orthogonal modulation systems (i.e., digital communications in which the alphabet consists of orthogonal waveforms) are known to result in good power efficiency at the expense of poor bandwidth utilization.[1,2] As the alphabet size M is increased, the energy-per-bit E required to achieve a given error probability P_e diminishes, but the information rate to bandwidth ratio (R/W) diminishes even more rapidly. Biorthogonal and simplex modulation afford somewhat improved performance, but are likewise restricted to low values of R/W.

There is considerable interest in finding large alphabet systems which have both good power efficiency and good bandwidth utilization.

Slepian[3] has given bounds on what can be achieved, but constructive techniques for approaching these bounds are generally unknown.

Although computer evaluation is ultimately required for precise knowledge of error probability, it is of interest to consider simpler performance criteria which permit at least a qualitative comparison of various systems without extensive computation. It is the purpose of this paper to demonstrate the utility of the latter approach.

After defining the problem more precisely in Section II, some well-known bounds on the error probability are employed in Section III to obtain a simple analytic criterion for comparing systems in the limit of low P_e. In Section IV this criterion is applied to systems (PSK, FSK, biorthogonal, and simplex) for which extensive exact computations are available and for which the conclusions drawn are already well-known. After these illustrative examples, permutation modulation[4] is considered in Section V and N-orthogonal phase modulation[5,6] in Section VI. It is shown that the former can yield better performance than conventional techniques, but that the latter is generally inferior. Finally, in Section VII limits on our performance criterion, obtained from sphere-packing arguments, are presented.

II. COMMUNICATION SYSTEM MODEL

We consider an M-ary modulation system of equienergy waveforms $S_i(t)$, $i = 1, \cdots, M$, on $(0,T)$, having the correlation matrix

$$\rho_{ij} = \frac{1}{E_s} \int_0^T S_i(t) S_j(t)\, dt. \tag{1}$$

E_s is the energy of each waveform so that $\rho_{ii} = 1$, and $-1 \leqq \rho_{ij} \leqq 1$. It is conventional[3,7] to define a normalized information rate, $(2 \log_2 M)/n$, where $n \leqq M$ is the rank of the correlation matrix (dimensionality of the signal space). We choose to call this normalized rate the "information to bandwidth ratio, R/W" motivated by the relations

$$\frac{R}{W} = \frac{\log_2 M}{WT} = \frac{2 \log_2 M}{n}, \tag{2}$$

where the second equality follows if we set $n = 2TW$, which is at least partially justified for large n by the work of Pollak and Landau.[8] For our purposes, the right-hand side of (2) may be considered as the definition of R/W.

It will be assumed that in addition to $\rho_{ii} = 1$ that each row of the

correlation matrix can be written as a permutation of the first row. Considering the waveforms as vectors in an n-dimensional linear vector space, this means that each waveform sees an identical environment of neighboring waveforms. This restriction is a desirable one if it is desired to transmit each waveform with equal *a priori* probability. The restriction is satisfied by the various modulation systems mentioned in the introduction.* Slepian[9] has termed such systems "group codes for the Gaussian channel."

It is assumed that the receiver observes a waveform $z(t)$ on the interval $(0,T)$

$$z(t) = S_i(t) + n(t), \tag{3}$$

where $n(t)$ is a sample function from a wbite Gaussian noise process of spectral density N_o; i.e.,

$$\langle n(t)n(t')\rangle = \frac{N_o}{2}\,\delta(t' - t). \tag{4}$$

On the basis of this observation we wish to decide with minimum probability of error (P_e) which of the M waveforms was transmitted. The optimum (minimum P_e) receiver is known[10] to consist of M matched filters which give

$$z_j = \int_0^T z(t)S_j(t)\,dt = E_s(\rho_{ij} + x_j), \tag{5}$$

where

$$x_j = \frac{1}{E_s}\int_0^T n(t)S_j(t)\,dt \tag{6}$$

and decision that the kth waveform was transmitted is made if $z_k > z_j$ for all $j \neq k$; i.e., the decision is made on the basis of the largest matched filter output.

From (6), the x_j are zero-mean Gaussian variates with covariance

$$\begin{aligned}\langle x_j x_k\rangle &= \frac{1}{E_s^2}\int_0^T\int_0^T dt\,dt'\,\langle n(t)n(t')\rangle S_j(t)S_k(t') \\ &= \frac{N_o}{2E_s}\,\rho_{jk}\,. \end{aligned}\tag{7}$$

* The only commonly employed M-ary system (known to the author) which does not satisfy this restriction is M-level amplitude modulation.

Tbe error probability of this system is given by*

$$P_e = 1 - \int_{\Omega_i} \cdots \int dx_1 \cdots dx_M \, p(x_1 , \cdots x_M), \tag{8}$$

where $p(x_1 , \cdots , x_M)$ is the multi-variate zero-mean Gaussian distribution with covariance given by (7), and the region of integration Ω_i is defined by the condition

$$\Omega_i = \text{region in which } 1 + x_i > \rho_{ij} + x_j \text{ for all } j \neq i.$$

Clearly P_e is a function of M parameters: E_s/N_o, ρ_{12}, ρ_{13}, $\cdots$, ρ_{1M}, the first of which is a signal-to-noise ratio, the remainder of which describe the correlation properties of tbe modulation system.

Landau and Slepian[11] bave proved the long-conjectured result that P_e is minimized for a given M (but n unrestricted) by the simplex configuration in which the correlation matrix has the form†

$$\rho_{ij} = \begin{cases} 1 & i = j \\ -\dfrac{1}{M-1} & i \neq j \end{cases} \quad \text{simplex.} \tag{9}$$

The rank of this matrix is $n = M - 1$ so that

$$(R/W)_{\text{simplex}} = \frac{2 \log_2 M}{M - 1}. \tag{10}$$

For this case the expression for P_e may be reduced to a single integral[10] and numerical results are readily obtained.[13]

Weber[14] has derived locally optimum configurations when $M/2 \leq n \leq M - 1$. For $n = M/2$ a local optimum is the biorthogonal configuration in which the signal vectors are located along the coordinate axes (+ and −) of the n-dimensional vector space such that

$$\rho_{ij} = \begin{cases} 1 & i = j \\ -1 & i = j - (-1)^i \\ 0 & i \neq j,\, j - (-1)^i \end{cases} \quad \text{biorthogonal.} \tag{11}$$

* Actually this is the error probability assuming the ith signal is transmitted. However, under the assumption of equal *a priori* transmission of all signals, and the permutation property assumed for the correlation matrix, this probability is independent of i and is equal to the system probability of error.

† The "local optimality" of the simplex configuration (*viz.*, that P_e has a local minimum) had been proved previously by Balakrishnan.[12]

The rank of this $M \times M$ matrix is $M/2$ so that

$$\left(\frac{R}{W}\right)_{\text{biorthogonal}} = \frac{4 \log_2 M}{M}. \tag{12}$$

In this case P_e may also be expressed as a single integral which is readily evaluated by machine techniques. Although for a given value of M, biorthogonal modulation requires slightly more energy-per-bit to achieve a given P_e than simplex,* it is noted that (for large M) R/W for biorthogonal is essentially twice that of simplex. Furthermore, for biorthogonal half of the waveforms are the negatives of the remaining half; consequently, $M/2$ rather than M matched filters are required. For these reasons biorthogonal is generally preferred to simplex, and indeed has been employed for deep-space communications.[15]

The disadvantage of both simplex and biothogonal modulation is that good power efficiency is associated with large values of M (as it must be for any modulation system) which from (10) and (12) imply small values of R/W. Weber's[14] results indicate locally optimum systems with R/W between simplex and biorthogonal, but these are then also restricted to relatively small R/W.

Optimum systems (in the sense of minimum P_e) are not known for $n < M/2$. However, bounds on the error probability of optimum systems have been obtained[7] and evaluated.[8] (The upper bound is obtained by random coding arguments, and the lower bound by sphere packing arguments.) These bounds are extremely useful in assessing the performance of specific systems; however, to do so involves explicit evaluation of P_e for the specific systems of interest. This is at best a difficult numerical task. Furthermore, we may find in comparing two systems that one is better if we are interested in $P_e \approx 10^{-3}$, whereas the reverse is true when $P_e \approx 10^{-6}$. Also, in comparing systems with different values of M it may be unrealistic to compare P_e, since P_e is the word error probability, and the systems contain a different number of bits per word. Comparison on the basis of bit error probability involves a difficult conversion from word to bit error probability which involve coding arguments separate from the modulation system performance.[16] For all of these reasons it is desirable to find a simpler criterion than P_e which permits at least a gross comparison of modulation systems.

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* If the comparison is made for a fixed R/W rather than M then biorthogonal requires less energy per bit. The simple unqualified statement that simplex is the optimum modulation system is misleading.

III. BOUNDS ON ERROR PROBABILITY

One approach to comparing modulation systems is to obtain lower and upper bounds on the true error probability*

$$P_l \leqq P_e \leqq P_u \tag{13}$$

and to say that system 1 is better than system 2 if $P_{u1} < P_{l2}$.

If two systems are close in performance, the above procedure may not enable us to determine which is better unless the bounds are close. On the other hand, close bounds may be difficult to evaluate and may not lead to a simple performance criterion. We adopt the viewpoint here that it is desirable to have bounds, which although quite loose, lead to a simple sufficient condition for determining when one system is better than another.

Let

$$\rho = \max_{j \neq i} \rho_{ij} = \max_{j > 1} \rho_{1j} \,. \tag{14}$$

That is, ρ is the largest non-diagonal entry of the correlation matrix. It is readily established that†

$$\Phi\left(-\sqrt{\frac{E_s}{N_o}(1-\rho)}\right) \leqq P_e \leqq (M-1)\Phi\left(-\sqrt{\frac{E_s}{N_o}(1-\rho)}\right), \tag{15}$$

where

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} dy \exp(-y^2/2). \tag{16}$$

The lower bound is obtained by observing that P_e for an M-ary system can be no less than that of the binary system containing nearest neighbor waveforms. The upper bound follows from

$$P_e \leqq \sum_{j=2}^{M} \Phi\left(-\sqrt{\frac{E_s}{N_o}(1-\rho_{ij})}\right) \leqq (M-1)\Phi\left(-\sqrt{\frac{E_s}{N_o}(1-\rho)}\right) \tag{17}$$

where the first inequality in (17) is a consequence of the symmetry property of the system and the fact that the probability of a union of events is less than the sum of the probabilities of the events. The

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* We consider here bounds on word error probability, which, however, may be easily converted to bounds on bit error probability. For example, if a word is in error at least one bit is in error, and at most all bits are in error. Hence, $P_l/\log_2 M$ and P_u are lower and upper bounds on the bit error probability.

† These bounds are generally well known and appear widely in the literature; e.g., Refs. 7, 17, 18, 19. Also, as noted in the previous footnote, these bounds on word error probability are readily converted into bounds on bit error probability.

second inequality in (17) follows simply by observing that a sum of $(M - 1)$ terms is no greater than $(M - 1)$ times the largest term.

In comparing modulation systems with different alphabet size it is more appropriate to consider the energy per bit E rather than the signal energy E_s, where

$$E = E_s/\log_2 M. \tag{18}$$

Indeed, the parameter E/N_o is an appropriate measure of the power efficiency of a modulation system. The Shannon channel capacity formula requires that $E/N_o > \log_e 2$ to achieve arbitrarily small P_e, conventional systems generally require values of E/N_o at least 4 times the Shannon minimum.[2]

In terms of the parameter E/N_o the error probability bounds may be rewritten

$$\Phi\left(-\sqrt{\frac{E}{N_o}K}\right) \leqq P_e \leqq (M - 1)\Phi\left(-\sqrt{\frac{E}{N_o}K}\right), \tag{19}$$

where

$$K = (1 - \rho)\log_2 M. \tag{20}$$

We will say that system 1 is "better" than system 2 if

$$(M_1 - 1)\Phi\left(-\sqrt{\frac{E}{N_o}K_1}\right) < \Phi\left(-\sqrt{\frac{E}{N_o}K_2}\right). \tag{21}$$

Several conclusions are apparent from (21).

(*i*) In the limit of large E/N_o, $K_1 > K_2$ is sufficient to ensure that system 1 is better than system 2. (If $K_1 > K_2$ we will say that system 1 is "asymptotically better" than system 2.)

(*ii*) If system 1 is asymptotically better than system 2, then there exists a value of E/N_o above which system 1 is better than system 2. Below this value of E/N_o our formalism is generally inadequate to determine which system is better. (The critical value of E/N_o may be obtained by replacing the inequality in (21) by an equality.)

(*iii*) A binary system that is asymptotically better than an M-ary system is always better than the M-ary system.

Thus, we can always determine quite simply which of two systems is asymptotically better, and may, in many special cases, be able to make comparisons at specific E/N_o of interest.

It should be emphasized that the above comparison is on the basis

of the P_e obtained with the two systems when operated at the same average power and information rate. To complete the comparison, the bandwidth requirements of the two systems should also be considered. Thus, the parameter R/W, as well as K should be used in comparing systems.

In the following sections of this paper specific systems will be considered and represented by points on a K, R/W plot. This will enable an immediate comparison of the asymptotic performance of systems having the same R/W. It should be noted that Gilbert[17] used a similar plot in his 1952 paper which addressed the same subject considered here. Gilbert employed a (SNR, R/W) plot in which the effective signal-to-noise ratio (SNR) was obtained for a given P_e by using the upper bound in (19). Since tbe SNR is related to our $1/K$, better systems correspond to smaller SNR. Our purpose in writing this paper is *not* to argue that our plot is a better way to present the results than Gilbert's. (Indeed, since in general, P_e is much closer to the upper than to the lower bound, his method of comparison is somewhat better, although somewhat less convenient to use.) Our purpose rather is to resurrect these old methods which have been largely discarded since the advent of high-speed computation, and to illustrate their applicability to recently proposed modulation systems.

IV. PHASE, FREQUENCY, BIORTHOGONAL AND SIMPLEX MODULATIONS

4.1 *Phase-Shift Modulation*

For M phasors uniformly spaced on the unit circle, $\rho = \cos 2\pi/M$. Therefore,

$$K = 2 \log_2 M \sin^2 \frac{\pi}{M}. \tag{22}$$

Note that $K = 2$ for both $M = 2$ and $M = 4$* and falls off thereafter.

Since the dimensionality of the signal space is $n = 1$ for $M = 2$ and $n = 2$ for $M > 2$, it follows that R/W is given by

$$\frac{R}{W} = \begin{cases} 2 & \text{for} \quad M = 2, \\ \log_2 M & \text{for} \quad M > 2. \end{cases} \tag{23}$$

* K is maximized (for integer M) when $M = 3$. In practice, it is generally desirable to consider only those values of M which are integer powers of 2 (i.e., each symbol conveys an integer number of bits). We shall restrict our numerical examples to such cases.

TABLE I—PHASE-SHIFT MODULATION

M	K	R/W
2	2	2
4	2	2
8	0.88	3
16	0.30	4
32	0.098	5
64	0.030	6

Table I lists the K and R/W values for phase-shift modulation, and these are denoted by dots in Fig. 1. It is apparent that $M = 2$ and $M = 4$ are asymptotically better than the higher-order systems, and from our previous results this implies that the binary system is always better than the general M-ary case with $M > 4$.* Recall that we are

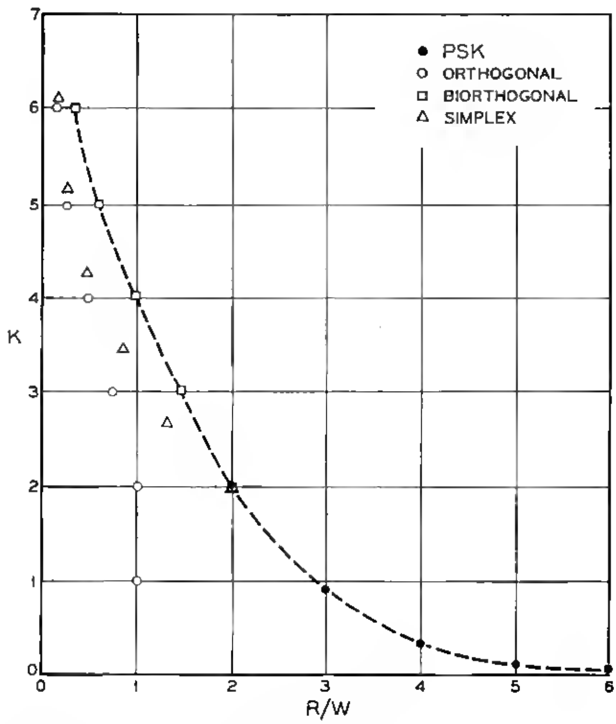

Fig. 1 — $(K,R/W)$ plot for phase-shift, orthogonal, biorthogonal, and simplex modulations.

consistently using the term "better" to mean smaller P_e for a given E/N_0. Large alphabet phase modulation may still be desirable because of the larger R/W.

4.2 *Frequency Shift (Orthogonal) Modulation*

For M orthogonal signals (e.g., frequency-shifted signals with essentially non-overlapping spectra), $\rho = 0$ and

$$K = \log_2 M. \tag{24}$$

The dimensionality of the signal space is the number of orthogonal vectors, $n = M$, so that

$$\frac{R}{W} = \frac{2 \log_2 M}{M}. \tag{25}$$

Table II lists the K and R/W values for orthogonal modulation, and these are denoted by circles in Fig. 1. Larger values of M correspond to systems which are asymptotically better, at the expense, however, of smaller values of R/W. It is clear that binary orthogonal is inferior to binary and quarternary PSK both in terms of a smaller K and smaller R/W.†

TABLE II—ORTHOGONAL MODULATION

M	K	R/W
2	1	1
4	2	1
8	3	3/4
16	4	1/2
32	5	5/16
64	6	3/16

4.3 *Biorthogonal Modulation*

A biorthogonal system consists of $M/2$ orthogonal waveforms and their negatives. The maximum correlation coefficient is $\rho = 0$ for $M \geqq 4$, but $\rho = -1$ for $M = 2$. Therefore,

$$K = \begin{cases} 2 & \text{for} \quad M = 2, \\ \log_2 M & \text{for} \quad M \geqq 4 \quad (M \text{ even}). \end{cases} \tag{26}$$

* This conclusion is confirmed by the exact calculations of P_e for M-ary PSK by C. R. Cahn.[20]

† Binary FSK may still be employed, of course, for simplicity reasons or because the channel phase coherence may not be consistent with phase-shift modulation.

Since $n = M/2$,

$$\frac{R}{W} = \frac{4 \log_2 M}{M}. \tag{27}$$

Table III lists the K and R/W values for biorthogonal modulation, and these are denoted by □'s in Fig. 1. Note that $M = 2$ and $M = 4$ biorthogonal are equivalent, respectively, to binary and quarternary PSK.

Clearly, for fixed R/W, biorthogonal is asymptotically better than orthogonal. For example, consider $M = 4$ orthogonal and $M = 16$ biorthogonal, both of which have $R/W = 1$. From (21), the biorthogonal system is better than the orthogonal system for all $E/N_o > 2.5$, which corresponds to all P_e of practical interest. ($P_e < 3(10)^{-2}$).

4.4 *Simplex Modulation*

In simplex modulation, the M code vectors form a regular simplex in $M - 1$ dimensions. (All vectors are equally spaced from all other vectors. This corresponds to an equilateral triangle in two dimensions, and a regular tetrahedron in three dimensions.) All correlation coefficients are equal and are given by[10,12,13] $\rho = -1/(M - 1)$. Therefore,

$$K = \frac{M}{M - 1} \log_2 M. \tag{28}$$

Since $n = M - 1$,

$$\frac{R}{W} = \frac{2 \log_2 M}{M - 1}. \tag{29}$$

Comparison of (28), (29) with (24), (25) indicates that for large M simplex modulation is essentially identical to orthogonal modulation. Table IV lists the K and R/W values for simplex modulation, and these are denoted by △'s in Fig. 1. A quick glance at Fig. 1 indicates

TABLE III—BIORTHOGONAL MODULATION

M	K	R/W
2	2	2
4	2	2
8	3	3/2
16	4	1
32	5	5/8
64	6	3/8

that depending on the R/W of interest, biorthogonal or PSK modulation offers the best asymptotic performance of the systems considered so far. (The dashed line in Fig. 1 is drawn through these "best" points.) Note that although simplex provides the largest K for a fixed value of M, it does not do so for fixed R/W.*

TABLE IV—SIMPLEX MODULATION

M	K	R/W
2	2	2
4	2.67	1.33
8	3.43	0.86
16	4.26	0.53
32	5.16	0.32
64	6.10	0.19

V. PERMUTATION MODULATION

Slepian[4] has recently described an exceedingly general modulation system (permutation modulation) for which all of the systems considered in the previous section are special cases. The optimum demodulation algorithm is particularly simple, but the actual evaluation of P_e, and the finding of good special cases is somewhat more complex. We restrict ourselves here to a special subclass of permutation modulation. This subclass is suggested both as the simplest generalization of biorthogonal systems, and because perusal of Slepian's results indicate that systems taken from this subclass are amongst the better of the moderate-sized alphabet examples which he considers.

Following Slepian we define an (n,m) permutation modulation system as follows. The time interval T is divided into n subintervals ($n = 2TW$). The first waveform of the alphabet consists of a signal with amplitude unity in the first m subintervals ($m < n$), and zero amplitude in the remaining subintervals. The remainder of the waveforms consist of all possible permutations of the subintervals, allowing also all combinations of plus and minus amplitudes. For example, the (3,2) system contains twelve waveforms which we may represent as

$$(1,1,0),\ (1,-1,0),\ (-1,1,0),\ (-1,-1,0),$$

$$(1,0,1),\ (1,0,-1),\ (-1,0,1),\ (-1,0,-1),$$

$$(0,1,1),\ (0,1,-1),\ (0,-1,1),\ (0,-1,-1).$$

* For the special case $M = 2$, simplex, biorthogonal and PSK are all equivalent.

In general, it is easily seen that the alphabet size M is given by

$$M = 2^m \binom{n}{m}. \tag{30}$$

It is also noted that the special case $(n,1)$ corresponds to biorthogonal modulation.*

This (n,m) modulation clearly satisfies the symmetry requirements of our theory. All members of the alphabet have equal energy† and the correlation matrix has the desired permutation property. It is readily seen that the maximum correlation coefficient is given by

$$\rho = \frac{m-1}{m}. \tag{31}$$

Thus,

$$\begin{aligned} K &\equiv (\log_2 M)(1-\rho) \\ &= 1 + \frac{1}{m}\log_2 \binom{n}{m} \end{aligned} \tag{32}$$

so that (n,m) modulation always achieves $K > 1$. Also

$$\begin{aligned} \frac{R}{W} &\equiv \frac{2\log_2 M}{n} \\ &= \frac{2m}{n} K. \end{aligned} \tag{33}$$

Equations (32) and (33) suggest that (n,m) modulation may achieve both large values of K and large R/W, which was not possible with any of the systems described in the previous section.

5.1 $(n,2)$ *Modulation*

Since $m = 1$ leads to biorthogonal modulation which has many desirable properties, it is natural to look next at the special case $m = 2$. From (32) and (33) it follows that for $(n,2)$,

$$K = \tfrac{1}{2}[1 + \log_2 n(n-1)] \qquad (n \geqq 3) \tag{34}$$

and

$$\frac{R}{W} = \frac{4K}{n}. \tag{35}$$

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* In Slepian's terminology, the (n,m) modulation described here is a variant II system in which $m_1 = n - m$, $m_2 = m$ and $\mu_1 = 0$, $\mu_2 = 1$.

† With the normalization employed above, the signal energy is m. However, all code words may be multiplied by a constant to achieve any desired E_s.

values of the K and R/W are given in Table V and are plotted as ■'s in Fig. 2. (For reference, Fig. 2 also contains the biorthogonal and PSK results from Fig. 1.) Thus, similar to biorthogonal, as n becomes large K increases but R/W decreases. It is seen from Fig. 2 that $(n,2)$ modulation gives better performance (larger K for a given R/W) than biorthogonal or PSK.*

5.2 *(2m,m) Modulation*

(2,1) corresponds to $M = 4$ biorthogonal, which from our earlier results gives $K = 2$, $R/W = 2$. It is seen from Table V that (4,2) gives $K = 2.30$, $R/W = 2.30$ which corresponds to both better asymptotic performance and better bandwidth utilization. It is apparent from (33) that whenever $n = 2m$, $R/W = K$, and an immediate question is how large can we make these two quantities.

TABLE V — $(n,2)$ MODULATION

n	$M = 2n(n-1)$	K	R/W
3	12	1.79	2.38
4	24	2.30	2.30
5	40	2.66	2.13
6	60	2.95	1.97
7	84	3.19	1.82

With $n = 2m$, it follows from (32) that

$$K = 1 + \frac{1}{m} \log_2 \binom{2m}{m}. \tag{36}$$

Use of Stirling's approximation when $m \gg 1$ gives

$$\binom{2m}{m} \approx \frac{1}{\sqrt{\pi m}} 2^{2m} \tag{37}$$

so that for large m, $K \to 3$. It is easily shown that K increases monotonically towards this asymptotic value as m is increased. Thus, $(2m, m)$ modulation does not permit attainment of arbitrarily large values of K, and hence cannot attain arbitrarily low P_e with finite E/N_o. This is consistent with Slepian's statement[4] that permutation modulation cannot approach channel capacity arbitrarily closely at non-zero R/W.

* This is, of course, achieved only at the expense of a larger alphabet size. It may also be noted from Table V that the alphabet size is not generally a power of 2 which may also be a practical disadvantage.

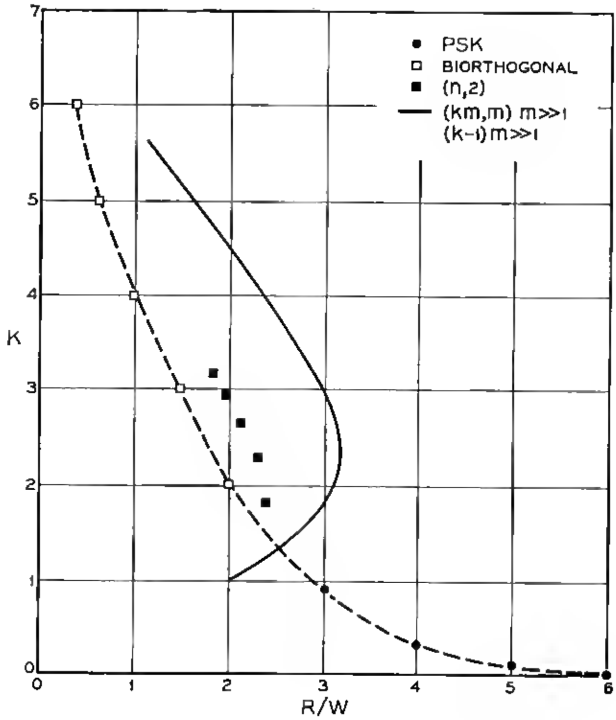

Fig. 2 — $(K, R/W)$ plot for permutation modulation.

5.3 *(km,m) Modulation (k > 1)*

As an immediate generalization of the above, consider the more general case $n = km$ where $k > 1$.* Then, from (33)

$$\frac{R}{W} = \frac{2}{k} K \tag{38}$$

and from (32)

$$K = 1 + \frac{1}{m} \log_2 \binom{km}{m}. \tag{39}$$

Again using Stirling's approximation for large m, (assuming also that $(k - 1)\, m \gg 1$)

$$\binom{km}{m} \approx \frac{1}{\sqrt{2\pi m}} \sqrt{\frac{k}{k-1}} \left(\frac{k}{k-1}\right)^{km} (k-1)^m. \tag{40}$$

* Of course k should be chosen so that km is an integer.

Thus, in the limit of large m, for fixed $k > 1$,

$$K \rightarrow 1 + k \log_2 \left(\frac{k}{k-1}\right) + \log_2 (k - 1). \tag{41}$$

For large k, the right-hand-side of (41) increases as log k; however, as seen from (38) R/W diminishes as k^{-1}. The locus of K, R/W values obtained with different values of k (but large m so that the approximation (41) applies) is shown by the solid curve in Fig. 2. In the limit as $k \rightarrow 1$ (but m always sufficiently large such that $(k - 1)\, m \gg 1$), $K \rightarrow 1$ and $R/W \rightarrow 2$. As k increases, both R/W and K increase until $k \approx 1.5$ at which point $R/W \approx 3.2$ and $K = 2.3$. Further increases in k result in a reduction in R/W but continued increase in K.

The above results indicate that (n,m) codes can be found with R/W as large as octary PSK ($R/W = 3$) and with considerably better asymptotic performance.

VI. COMBINED PHASE-SHIFT AND ORTHOGONAL MODULATION

In the previous examples we have compared by approximate methods modulation systems which have already been analyzed exactly. Although perhaps additional insight into the relative performance of these systems has been obtained, many of our conclusions may be inferred from existing exact calculations. We now wish to consider a new system, recently proposed by Reed and Scholtz,[5,6] which (to our knowledge) has not yet been evaluated numerically.

Consider an alphabet M divided into M_f groups, each group containing M_p members. Thus,

$$M = M_f M_p \,. \tag{42}$$

The different groups may be considered to be sufficiently separated in frequency so that waveforms from different groups are orthogonal. Within a group the waveforms have the correlation properties associated with phase-shift modulation. Thus for $M_p \geqq 4$ the maximum correlation coefficient is $\rho = \cos (2\pi/M_p)$, and

$$K = 2 \sin^2 \frac{\pi}{M_p} (\log_2 M_p + \log_2 M_f). \tag{43}$$

Since each group requires a two-dimensional sub-space

$$n = 2M_f \,. \tag{44}$$

Thus,

$$R/W = \frac{\log_2 M_f}{M_f} + \frac{\log_2 M_p}{M_f}. \tag{45}$$

In the special case of $M_f = 1$ it is apparent that this system reduces to simple phase-shift modulation (Section 4.1). In the special case of $M_p = 4$ it reduces to the biorthogonal case (Section 4.3). A question of interest then is whether choices of $M_p > 4$, $M_f > 1$ lead to better performance than either phase-shift or biorthogonal modulation.*

In Fig. 3 the K and R/W values (obtained from 43 and 45) are

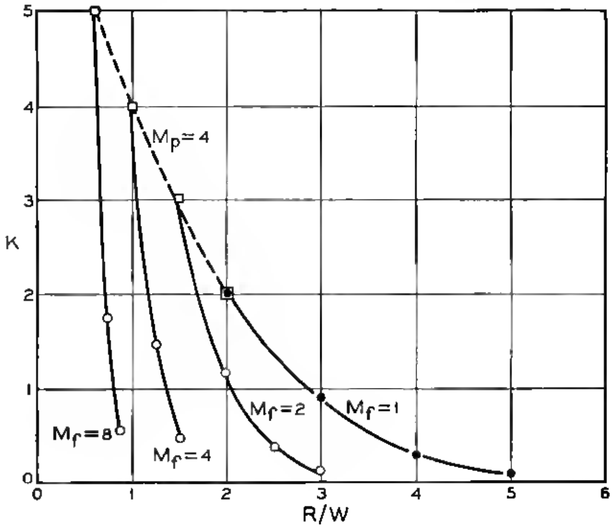

Fig. 3 — $(K,R/W)$ plot for combined phase-shift and orthogonal modulation.

shown for the combined phase orthogonal modulation. The solid curves are for constant values of M_f (noted on the curve); the uppermost point on each such curve corresponds to $M_p = 4$, and each lower point corresponds to M_p increased by a factor of two. The dashed curve goes through the $M_p = 4$ points (biorthogonal). It is apparent from this figure that in this class of systems, for $R/W \leqq 2$, the $M_p = 4$ biorthogonal systems give the largest value of K. For $R/W \geqq 2$, the $M_f = 1$ phase-shift systems give the largest value of K. Thus, *in terms*

* Reed and Scholtz[5,6] are concerned largely with an algebraic method of generating waveforms with the above correlation properties, rather than in a comparative evaluation of performance.

of asymptotic performance, choice of $M_p > 4$, $M_f > 1$ always gives poorer performance than systems which achieve the same R/W with either $M_p = 4$ (biorthogonal) or $M_f = 1$ (simple phase shift).

For example, consider $M_f = 2$, $M_p = 8$. This yields $R/W = 2$ and $K = 1.17$. However, $R/W = 2$ is also achieved with $M_f = 1$, $M_p = 4$ (quaternary PSK), and for this case $K = 2$. From (21) we can conclude that the latter system is better than the former for all $E/N_o > 2.5$, which includes all P_e of interest. The significance of these results is that we can make this comparison with only a simple slide-rule calculation.

In the above comparison we considered only $M_p \geqq 4$. The case of $M_p = 1$, $M_f > 1$ is the orthogonal modulation previously considered. The case of $M_p = 2$, $M_f > 1$ gives the same performance as biorthogonal but achieves only $\frac{1}{2}$ the R/W and consequently is of little interest. The case of $M_p = 3$, $M_f > 1$ consists of orthogonal combinations of two-dimensional simplexes (equilateral triangles). Reed and Scholtz[6] conjecture that for $M = 3M_f$, the three-phase orthogonal system gives a smaller P_e than any other collection of $3M_f$ signal functions in a space of dimensionality $2M_f$. Although this conjecture may well be true, we wish to point out that if the comparison is made on the basis of fixed R/W (rather than fixed M) then biorthogonal is asymptotically better than three-phase orthogonal. One way of seeing this is by noting that three-phase orthogonal has the same K but smaller R/W than the four-phase (biorthogonal) system of the same dimensionality. To increase the R/W of the three-phase system requires a reduction in K which makes it asymptotically poorer than the corresponding biorthogonal system.

VII. BOUNDS ON K

It has been shown that the $(K,R/W)$ plot provides a useful technique for comparing the performance of various modulation systems. Although our main concern here is in the comparison of specific systems, it is still natural to ask whether there are bounds on what may be achieved in the $(K,R/W)$ plane.

It is apparent from the definition of K

$$K = (1 - \rho) \log_2 M \tag{46}$$

that if no constraint is placed on alphabet size or signal space dimensionality, K can, in principle, be made arbitrarily large for any

R/W. This corresponds to the fact that the Shannon channel capacity formula implies that arbitrarily small P_e may be achieved at all (finite) R/W with finite E/N_o.

If M is held fixed but n is unconstrained, then the maximum K is achieved by the simplex modulation[12] (Section 4.4) for which case $K = [M/(M-1)] \log_2 M$ and $R/W = (2 \log_2 M)/M - 1$.

Perhaps of more practical interest is the opposite case where the signal space dimensionality n is fixed, but M is unconstrained. Here, sphere-packing arguments may be used to show that[3]

$$M \leqq 2/I_{(1-\rho)/2}\left(\frac{n-1}{2}, \frac{1}{2}\right), \tag{47}$$

where $I_x(p,q)$ is the incomplete beta-function which is extensively tabulated.[21] Thus, for a given ρ and n, an upper bound to M may be calculated from (47). Since $I_x(p,q)$ is monotonic increasing in x, this also gives a lower bound on ρ for fixed M and n. Considered in this latter context we can then determine an upper bound on K with which is associated a given value of $R/W = (2/n) \log_2 M$. This upper bound, K_u, is plotted in Fig. 4 as a function of R/W for $n = 5$ and $n = 10$. Both curves indicate that K_u achieves a maximum value. This is understandable since for large R/W, $1 - \rho$ decreases more rapidly than $\log_2 M$ increases. On the other hand, as R/W decreases, $\log_2 M$ keeps decreasing, whereas $1 - \rho$ is of course always less than 2. Thus, it is not surprising that there exists an R/W at which K_u is a maximum.

It should be noted, however, that K_u is an upper bound which likely cannot be achieved. For example, when $R/W = (2/n) \log_2 2n$, corresponding to $M = 2n$, the optimum configuration is widely conjectured to be the biorthogonal case.[14] The corresponding K and R/W values for biorthogonal with $n = 10$ and $n = 5$ are shown by the points marked (10,1) and (5,1) on the dashed curves of Fig. 4. These points lie well below the upper bounds.

Biorthogonal is a special case ($m = 1$) of the (n,m) permutation modulation considered in Section V. Fig. 4 (dashed curves) shows the K and R/W values for the $(10,m)$ and $(5,m)$ cases. As must be, these curves lie below the upper bounds given by the solid curves.

Finally, we note from Fig. 4 that (n,m) permutation modulation possesses the interesting feature that as m is increased (for a fixed n) a maximum R/W is achieved. Both the properties of the maxima of K_u and the maxima of the R/W of (n,m) modulation are probably worthy of further study.

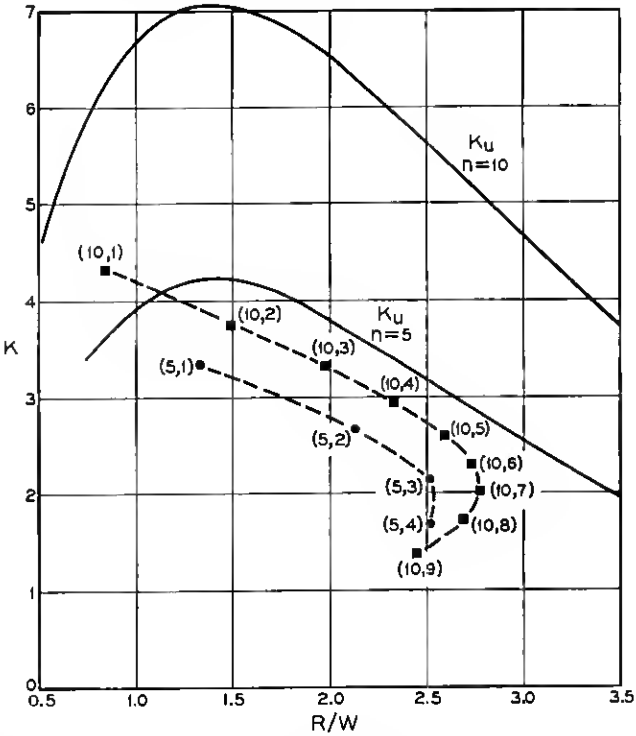

Fig. 4 — Bounds on $(K,R/W)$ for fixed n and comparison with permutation modulation.

VIII. CONCLUSION

The main conclusion to be drawn is that the K, R/W plot provides an exceedingly useful technique for comparing modulation systems. We have restricted ourselves to modulation systems in which the signal alphabet consists of equienergy waveforms for which all rows of the correlation matrix are permutations of a given row. (Geometrically, the alphabet consists of M points on the surface of an n-dimensional sphere such that all points see exactly the same environment.) This class of systems, although somewhat limited, is sufficiently broad to cover most systems of theoretical and practical interest. Given two systems in this class such that $K_1 > K_2$; then in the limit of large E/N_o (low P_e) $P_{e1} < P_{e2}$ for the same E/N_o. Furthermore, we have obtained a simple sufficient condition on the E/N_o above which this inequality is valid. These results are in reality not new.

They are implicit in the results of Shannon[7] and in many other works.[19] What is perhaps new is that many interesting results and comparisons can be obtained by such simple techniques.

Considerably more precise comparisons can of course be made by exact computation of P_e rather than by comparison of K. The latter procedure however is considerably quicker and allows ready consideration of entire classes of systems (e.g., the (n,m) permutation modulation and the combined phase-shift orthogonal modulations considered in the previous sections). The comparisons discussed here are not meant to supplant exact evaluation, but rather as a coarse sieve for delineating systems worthy of more extensive calculation.

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